An [abstract finite-dimensional complex Lie algebra](http://en.wikipedia.org/wiki/Lie_algebra), or *Lie algebra* for short, is a finite-dimensional complex vector space  together with an anti-symmetric bilinear form  that obeys the [Jacobi identity](http://en.wikipedia.org/wiki/Jacobi_identity)



for all ; by anti-symmetry one can also rewrite the Jacobi identity as



We will usually omit the subscript from the Lie bracket  when this will not cause ambiguity. A *homomorphism*  between two Lie algebras  is a linear map that respects the Lie bracket, thus  for all . As with many other classes of mathematical objects, the class of Lie algebras together with their homomorphisms then form a [category](http://en.wikipedia.org/wiki/Category_(mathematics)). One can of course also consider Lie algebras in infinite dimension or over other fields, but we will restrict attention throughout these notes to the finite-dimensional complex case. The trivial, zero-dimensional Lie algebra is denoted ; Lie algebras of positive dimension will be called *non-trivial*.

Lie algebras come up in many contexts in mathematics, in particular arising as the tangent space of complex [Lie groups](http://en.wikipedia.org/wiki/Lie_group). It is thus very profitable to think of Lie algebras as being the infinitesimal component of a Lie group, and in particular almost all of the notation and concepts that are applicable to Lie groups (e.g. nilpotence, solvability, extensions, etc.) have infinitesimal counterparts in the category of Lie algebras (often with exactly the same terminology). See [this previous blog post](https://terrytao.wordpress.com/2011/09/01/254a-notes-1-lie-groups-lie-algebras-and-the-baker-campbell-hausdorff-formula/) for more discussion about the connection between Lie algebras and Lie groups (that post was focused over the reals instead of the complexes, but much of the discussion carries over to the complex case).

A particular example of a Lie algebra is the general linear Lie algebra  of linear transformations  on a finite-dimensional complex vector space (or *vector space* for short) , with the commutator Lie bracket ; one easily verifies that this is indeed an abstract Lie algebra. We will define a *concrete* Lie algebra to be a Lie algebra that is a subalgebra of  for some vector space , and similarly define a *representation* of a Lie algebra  to be a homomorphism  into a concrete Lie algebra . It is a deep [theorem of Ado](http://en.wikipedia.org/wiki/Ado%27s_theorem) (discussed in [this previous post](https://terrytao.wordpress.com/2011/05/10/ados-theorem/)) that every abstract Lie algebra is in fact isomorphic to a concrete one (or equivalently, that every abstract Lie algebra has a faithful representation), but we will not need or prove this fact here.

Even without Ado’s theorem, though, the structure of abstract Lie algebras is very well understood. As with objects in many other algebraic categories, a basic way to understand a Lie algebra  is to factor it into two simpler algebras  via a [short exact sequence](http://en.wikipedia.org/wiki/Short_exact_sequence#Short_exact_sequence)



thus one has an injective homomorphism from  to  and a surjective homomorphism from  to  such that the image of the former homomorphism is the kernel of the latter. (To be pedantic, a short exact sequence in a general category requires these homomorphisms to be [monomorphisms](http://en.wikipedia.org/wiki/Monomorphism) and [epimorphisms](http://en.wikipedia.org/wiki/Epimorphism) respectively, but in the category of Lie algebras these turn out to reduce to the more familiar concepts of injectivity and surjectivity respectively.) Given such a sequence, one can (non-uniquely) identify  with the vector space  equipped with a Lie bracket of the form



for some bilinear maps  and  that obey some Jacobi-type identities which we will not record here. Understanding exactly what maps  are possible here (up to coordinate change) can be a difficult task (and is one of the key objectives of [Lie algebra cohomology](http://en.wikipedia.org/wiki/Lie_algebra_cohomology)), but in principle at least, the problem of understanding  can be reduced to that of understanding that of its factors . To emphasise this, I will (perhaps idiosyncratically) express the existence of a short exact sequence (3) by the [ATLAS](http://en.wikipedia.org/wiki/ATLAS_of_Finite_Groups)-type notation



although one should caution that for given  and , there can be multiple non-isomorphic  that can form a short exact sequence with , so that  is not a uniquely defined combination of  and ; one could emphasise this by writing  instead of , though we will not do so here. We will refer to  as an *extension* of  by , and read the notation (5) as “  is -by-“; confusingly, these two notations reverse the subject and object of “by”, but unfortunately both notations are well entrenched in the literature. We caution that the operation  is not commutative, and it is only partly associative: every Lie algebra of the form  is also of the form , but the converse is not true (see [this previous blog post](https://terrytao.wordpress.com/2010/01/23/some-notes-on-group-extensions/) for some related discussion). As we are working in the infinitesimal world of Lie algebras (which have an additive group operation) rather than Lie groups (in which the group operation is usually written multiplicatively), it may help to think of  as a (twisted) “sum” of  and  rather than a “product”; for instance, we have  and , and also .

Special examples of extensions  of  by  include the [direct sum](http://en.wikipedia.org/wiki/Direct_product) (or *direct product*)  (also denoted ), which is given by the construction (4) with  and  both vanishing, and the [split extension](http://en.wikipedia.org/wiki/Splitting_lemma) (or [semidirect product](http://en.wikipedia.org/wiki/Semidirect_product))  (also denoted ), which is given by the construction (4) with  vanishing and the bilinear map  taking the form



for some representation  of  in the concrete Lie algebra of [derivations](http://en.wikipedia.org/wiki/Derivation_(abstract_algebra))  of , that is to say the algebra of linear maps  that obey the Leibniz rule



for all . (The derivation algebra  of a Lie algebra  is analogous to the [automorphism group](http://en.wikipedia.org/wiki/Automorphism_group)  of a Lie group , with the two concepts being intertwined by the tangent space functor  from Lie groups to Lie algebras (i.e. the derivation algebra is the infinitesimal version of the automorphism group). Of course, this functor also intertwines the Lie algebra and Lie group versions of most of the other concepts discussed here, such as extensions, semidirect products, etc.)

There are two general ways to factor a Lie algebra  as an extension  of a smaller Lie algebra  by another smaller Lie algebra . One is to locate a [Lie algebra ideal](http://en.wikipedia.org/wiki/Lie_algebra#Homomorphisms.2C_subalgebras.2C_and_ideals) (or *ideal* for short)  in , thus , where  denotes the Lie algebra generated by , and then take  to be the quotient space  in the usual manner; one can check that ,  are also Lie algebras and that we do indeed have a short exact sequence



Conversely, whenever one has a factorisation , one can identify  with an ideal in , and  with the quotient of  by .

The other general way to obtain such a factorisation is is to start with a homomorphism  of  into another Lie algebra , take  to be the image  of , and  to be the kernel . Again, it is easy to see that this does indeed create a short exact sequence:



Conversely, whenever one has a factorisation , one can identify  with the image of  under some homomorphism, and  with the kernel of that homomorphism. Note that if a representation  is [faithful](http://en.wikipedia.org/wiki/Faithful_representation) (i.e. injective), then the kernel is trivial and  is isomorphic to .

Now we consider some examples of factoring some class of Lie algebras into simpler Lie algebras. The easiest examples of Lie algebras to understand are the *abelian* Lie algebras , in which the Lie bracket identically vanishes. Every one-dimensional Lie algebra is automatically abelian, and thus isomorphic to the scalar algebra . Conversely, by using an arbitrary linear basis of , we see that an abelian Lie algebra is isomorphic to the direct sum of one-dimensional algebras. Thus, a Lie algebra is abelian if and only if it is isomorphic to the direct sum of finitely many copies of .

Now consider a Lie algebra  that is not necessarily abelian. We then form the [derived algebra](http://en.wikipedia.org/wiki/Derived_subgroup) ; this algebra is trivial if and only if  is abelian. It is easy to see that  is an ideal whenever  are ideals, so in particular the derived algebra  is an ideal and we thus have the short exact sequence



The algebra  is the maximal abelian quotient of , and is known as the [abelianisation](http://en.wikipedia.org/wiki/Derived_subgroup#Abelianization) of . If it is trivial, we call the Lie algebra [perfect](http://en.wikipedia.org/wiki/Perfect_group). If instead it is non-trivial, then the derived algebra has strictly smaller dimension than . From this, it is natural to associate two series to any Lie algebra , the *lower central series*



and the *derived series*



By induction we see that these are both decreasing series of ideals of , with the derived series being slightly smaller ( for all ). We say that a Lie algebra is [nilpotent](http://en.wikipedia.org/wiki/Nilpotent_Lie_algebra) if its lower central series is eventually trivial, and [solvable](http://en.wikipedia.org/wiki/Solvable_Lie_algebra) if its derived series eventually becomes trivial. Thus, abelian Lie algebras are nilpotent, and nilpotent Lie algebras are solvable, but the converses are not necessarily true. For instance, in the general linear group , which can be identified with the Lie algebra of  complex matrices, the subalgebra  of strictly upper triangular matrices is nilpotent (but not abelian for ), while the subalgebra  of upper triangular matrices is solvable (but not nilpotent for ). It is also clear that any subalgebra of a nilpotent algebra is nilpotent, and similarly for solvable or abelian algebras.

From the above discussion we see that a Lie algebra is solvable if and only if it can be represented by a tower of abelian extensions, thus



for some abelian . Similarly, a Lie algebra  is nilpotent if it is expressible as a tower of *central* extensions (so that in all the extensions  in the above factorisation,  is central in , where we say that  is central in  if ). We also see that an extension  is solvable if and only of both factors  are solvable. Splitting abelian algebras into cyclic (i.e. one-dimensional) ones, we thus see that a finite-dimensional Lie algebra is solvable if and only if it is [polycylic](http://en.wikipedia.org/wiki/Polycyclic_group), i.e. it can be represented by a tower of cyclic extensions.

For our next fundamental example of using short exact sequences to split a general Lie algebra into simpler objects, we observe that every abstract Lie algebra  has an [adjoint representation](http://en.wikipedia.org/wiki/Adjoint_representation) , where for each ,  is the linear map ; one easily verifies that this is indeed a representation (indeed, (2) is equivalent to the assertion that  for all ). The kernel of this representation is the [center](http://en.wikipedia.org/wiki/Center_(group_theory)) , which the maximal central subalgebra of . We thus have the short exact sequence



which, among other things, shows that every abstract Lie algebra is a central extension of a concrete Lie algebra (which can serve as a cheap substitute for Ado’s theorem mentioned earlier).

For our next fundamental decomposition of Lie algebras, we need some more definitions. A Lie algebra  is [simple](http://en.wikipedia.org/wiki/Simple_Lie_algebra) if it is non-abelian and has no ideals other than  and ; thus simple Lie algebras cannot be factored  into strictly smaller algebras . In particular, simple Lie algebras are automatically perfect and centerless. We have the following fundamental theorem:

**Theorem 1 (Equivalent definitions of semisimplicity)** Let  be a Lie algebra. Then the following are equivalent:

* (i)  does not contain any non-trivial solvable ideal.
* (ii)  does not contain any non-trivial abelian ideal.
* (iii) The [Killing form](http://en.wikipedia.org/wiki/Killing_form) , defined as the bilinear form , is non-degenerate on .
* (iv)  is isomorphic to the direct sum of finitely many non-abelian simple Lie algebras.

We review the proof of this theorem later in these notes. A Lie algebra obeying any (and hence all) of the properties (i)-(iv) is known as a [semisimple](http://en.wikipedia.org/wiki/Semisimple_Lie_algebra) Lie algebra. The statement (iv) is usually taken as the *definition* of semisimplicity; the equivalence of (iv) and (i) is a special case of *Weyl’s complete reducibility theorem* (see Theorem 32), and the equivalence of (iv) and (iii) is known as the *Cartan semisimplicity criterion*. (The equivalence of (i) and (ii) is easy.)

If  and  are solvable ideals of a Lie algebra , then it is not difficult to see that the vector sum  is also a solvable ideal (because on quotienting by  we see that the derived series of  must eventually fall inside , and thence must eventually become trivial by the solvability of ). As our Lie algebras are finite dimensional, we conclude that  has a unique maximal solvable ideal, known as the [radical](http://en.wikipedia.org/wiki/Radical_of_a_Lie_algebra)  of . The quotient  is then a Lie algebra with trivial radical, and is thus semisimple by the above theorem, giving the [Levi decomposition](http://en.wikipedia.org/wiki/Levi_decomposition)



expressing an arbitrary Lie algebra as an extension of a semisimple Lie algebra  by a solvable algebra  (and it is not hard to see that this is the only possible such extension up to isomorphism). Indeed, a deep theorem of Levi allows one to upgrade this decomposition to a split extension



although we will not need or prove this result here.

In view of the above decompositions, we see that we can factor any Lie algebra (using a suitable combination of direct sums and extensions) into a finite number of simple Lie algebras and the scalar algebra . In principle, this means that one can understand an arbitrary Lie algebra once one understands all the simple Lie algebras (which, being defined over , are somewhat confusingly referred to as *simple complex Lie algebras* in the literature). Amazingly, this latter class of algebras are completely classified:

**Theorem 2 (Classification of simple Lie algebras)** Up to isomorphism, every simple Lie algebra is of one of the following forms:

*  for some .
*  for some .
*  for some .
*  for some .
* , or .
* .
* .

(The precise definition of the [classical Lie algebras](http://en.wikipedia.org/wiki/Classical_group)  and the exceptional Lie algebras  will be recalled later.)

(One can extend the families  of classical Lie algebras a little bit to smaller values of , but the resulting algebras are either isomorphic to other algebras on this list, or cease to be simple; see [this previous post](https://terrytao.wordpress.com/2011/03/11/exceptional-isogenies-between-the-classical-lie-groups/) for further discussion.)

This classification is a basic starting point for the classification of many other related objects, including Lie algebras and Lie groups over more general fields (e.g. the reals ), as well as finite simple groups. Being so fundamental to the subject, this classification is covered in almost every basic textbook in Lie algebras, and I myself learned it many years ago in an honours undergraduate course back in Australia. The proof is rather lengthy, though, and I have always had difficulty keeping it straight in my head. So I have decided to write some notes on the classification in this blog post, aiming to be self-contained (though moving rapidly). There is no new material in this post, though; it is all drawn from standard reference texts (I relied particularly on [Fulton and Harris’s text](http://www.ams.org/mathscinet-getitem?mr=1153249), which I highly recommend). In fact it seems remarkably hard to deviate from the standard routes given in the literature to the classification; I would be interested in knowing about other ways to reach the classification (or substeps in that classification) that are genuinely different from the orthodox route.

**— 1. Abelian representations —**

One of the key strategies in the classification of a Lie algebra  is to work with representations of , particularly the adjoint representation , and then restrict such representations to various simpler subalgebras  of , for which the representation theory is well understood. In particular, one aims to exploit the representation theory of *abelian* algebras (and to a lesser extent, nilpotent and solvable algebras), as well as the fundamental example of the two-dimensional [special linear Lie algebra](http://en.wikipedia.org/wiki/Special_linear_Lie_algebra) , which is the smallest and easiest to understand of the simple Lie algebras, and plays an absolutely crucial role in exploring and then classifying all the other simple Lie algebras.

We begin this program by recording the representation theory of abelian Lie algebras. We begin with representations  of the one-dimensional algebra . Setting , this is essentially the representation theory of a single linear transformation . Here, the theory is given by the [Jordan decomposition](http://en.wikipedia.org/wiki/Jordan%E2%80%93Chevalley_decomposition). Firstly, for each complex number , we can define the [generalised eigenspace](http://en.wikipedia.org/wiki/Generalized_eigenspace)



One easily verifies that the  are all linearly independent -invariant subspaces of , and in particular that there are only finitely many  (the *spectrum*  of ) for which  is non-trivial. If one quotients out all the generalised eigenspaces, one can check that the quotiented transformation  no longer has any spectrum, which contradicts the [fundamental theorem of algebra](http://en.wikipedia.org/wiki/Fundamental_theorem_of_algebra) applied to the [characteristic polynomial](http://en.wikipedia.org/wiki/Characteristic_polynomial) of this quotiented transformation (or, if is more analytically inclined, one could apply [Liouville’s theorem](http://en.wikipedia.org/wiki/Liouville%27s_theorem_(complex_analysis)) to the [resolvent operators](http://en.wikipedia.org/wiki/Resolvent_formalism) to obtain the required contradiction). Thus the generalised eigenspaces span :



On each space , the operator  only has spectrum at zero, and thus (again from the fundamental theorem of algebra) has non-trivial kernel; similarly for any -invariant subspace of , such as the range  of . Iterating this observation we conclude that  is a [nilpotent operator](http://en.wikipedia.org/wiki/Nilpotent_operator) on , thus  for some . If we then write  to be the direct sum of the scalar multiplication operators  on each generalised eigenspace , and  to be the direct sum of the operators  on these spaces, we have obtained the [Jordan decomposition](http://en.wikipedia.org/wiki/Jordan%E2%80%93Chevalley_decomposition) (or *Jordan-Chevalley decomposition*)



where the operator  is [semisimple](http://en.wikipedia.org/wiki/Semi-simple_operator) in the sense that it is a diagonalisable linear transformation on  (or equivalently, all generalised eigenspaces are actually eigenspaces), and  is nilpotent. Furthermore, as we may use polynomial interpolation to find a polynomial  such that  vanishes to arbitrarily high order at  for each  (and also ), we see that  (and hence ) can be expressed as polynomials in  with zero constant coefficient; this fact will be important later. In particular,  and  commute.

Conversely, given an arbitrary linear transformation , the Jordan-Chevalley decomposition is the unique decomposition into commuting semisimple and nilpotent elements. Indeed, if we have an alternate decomposition  into a semisimple element  commuting with a nilpotent element , then the generalised eigenspaces of  must be preserved by both  and , and so without loss of generality we may assume that there is just a single generalised eigenspace ; subtracting  we may then assume that , but then  is nilpotent, and so  is also nilpotent; but the only transformation which is both semisimple and nilpotent is the zero transformation, and the claim follows.

From the Jordan-Chevalley decomposition it is not difficult to then place  in [Jordan normal form](http://en.wikipedia.org/wiki/Jordan_normal_form) by selecting a suitable basis for ; see e.g. [this previous blog post](https://terrytao.wordpress.com/2007/10/12/the-jordan-normal-form-and-the-euclidean-algorithm/). But in contrast to the Jordan-Chevalley decomposition, the basis is not unique in general, and we will not explicitly use the Jordan normal form in the rest of this post.

Given an abstract complex vector space , there is in general no canonical notion of complex conjugation on , or of linear transformations . However, we can define the conjugate  of any *semisimple* transformation , defined as the direct sum of  on each eigenspace  of . In particular, we can define the conjugate  of the semisimple component  of an arbitrary linear transformation , which will be the direct sum of  on each *generalised* eigenspace  of . The significance of this transformation lies in the observation that the product  has trace  on each generalised eigenspace (since nilpotent operators have zero trace), and in particular we see that



if and only if the spectrum consists only of zero, or equivalently that  is nilpotent. Thus (7) provides a test for nilpotency, which will be turn out to be quite useful later in this post. (Note that this trick relies very much on the special structure of , in particular the fact that it has characteristic zero.)

In the above arguments we have used the basic fact that if two operators  and  commute, then the generalised eigenspaces of one operator are preserved by the other. Iterating this fact, we can now start understanding the representations  of an abelian Lie algebra. Namely, there is a finite set  of linear functionals (or homomorphisms)  on  (i.e. elements of the dual space ) for which the generalised eigenspaces



are non-trivial and -invariant, and we have the decomposition



Here we use  as short-hand for writing  for all . An important special case arises when the action of  is semisimple in the sense that  is semisimple for all . Then all the generalised eigenspaces are just eigenspaces (or [weight spaces](http://en.wikipedia.org/wiki/Weight_space#Weight_space_of_a_representation)) , thus



for all  and . When this occurs we call  a *weight vector* with weight .

**— 2. Engel’s theorem and Lie’s theorem —**

In the introduction we gave the two basic examples of nilpotent and solvable Lie algebras, namely the strictly upper triangular and upper triangular matrices. The theorems [of Engel](http://en.wikipedia.org/wiki/Engel%27s_theorem) and [of Lie](http://en.wikipedia.org/wiki/Lie%27s_theorem#Lie.27s_theorem) assert, roughly speaking, that these examples (and subalgebras thereof) are essentially the only type of solvable and nilpotent Lie algebras that can exist, at least in the concrete setting of subalgebras of . Among other things, these theorems greatly clarify the representation theory of nilpotent and solvable Lie algebras.

We begin with Engel’s theorem.

**Theorem 3 (Engel’s theorem)** Let  be a concrete Lie algebra such that every element  of  is nilpotent as a linear transformation on .

* (i) If  is non-trivial, then there is a non-zero element  of  which is annihilated by every element of .
* (ii) There is a basis of  for which all elements of  are strictly upper triangular. In particular,  is nilpotent.

*Proof:* We begin with (i). We induct on the dimension of . The claim is trivial for dimensions  and , so suppose that  has dimension greater than , and that the claim is already proven for smaller dimensions.

Let  be a maximal proper subalgebra of , then  has dimension strictly between zero and  (since all one-dimensional subspaces are proper subalgebras). Observe that for every ,  acts on both the vector spaces  and  and thus also on the quotient space . As  is nilpotent, all of these actions are nilpotent also. In particular, by induction hypothesis, there is  which is annihilated by  for all . Let  be a representative of  in , then , and so  is a subalgebra and is thus all of .

By induction hypothesis again, the space  of vectors in  annihilated by  is non-trivial; as , it is preserved by . As  is nilpotent, there is a non-trivial element of  annihilated by  and hence by , as required.

Now we prove (ii). We induct on the dimension of . The case of dimension zero is trivial, so suppose  has dimension at least one, and the claim has already been proven for dimension . By (i), we may find a non-trivial vector  annihilated by , and so we may project  down to . By the induction hypothesis, there is a basis for  on which the projection of any element of  is strictly upper-triangular; pulling this basis back to  and adjoining , we obtain the claim. 

As a corollary of this theorem and the short exact sequence (6) we see that an abstract Lie algebra  is nilpotent iff  is nilpotent iff  is nilpotent in  for every  (i.e. every element of  is *ad-nilpotent*).

Engel’s theorem is in fact valid over every field. The analogous theorem of Lie for solvable algebras, however, relies much more strongly on the specific properties of the complex field .

**Theorem 4 (Lie’s theorem)** Let  be a solvable concrete Lie algebra.

* (i) If  is non-trivial, there exists a non-zero element  of  which is an eigenvector for every element of .
* (ii) There is a basis for  such that every element of  is upper triangular.

Note that if one specialises Lie’s theorem to abelian  then one essentially recovers the abelian theory of the previous section.

*Proof:* We prove (i). As before we induct on the dimension of . The dimension zero case is trivial, so suppose that  has dimension at least one and that the claim has been proven for smaller dimensions.

Let  be a codimension one subalgebra of ; such an algebra can be formed by taking a codimension one subspace of the abelianisation  (which has dimension at least one, else  will not be solvable) and then pulling back to . Note that  is automatically an ideal.

By induction, there is a non-zero element  of  such that every element of  has  as an eigenvector, thus we have



for all  and some linear functional . If we then set  to be the simultaneous eigenspace



then  is a non-trivial subspace of .

Let  be an element of  that is not in , and let . Consider the space spanned by the orbit . By finite dimensionality, this space has a basis  for some . By induction and definition of , we see that every  acts on this space by an upper-triangular matrix with diagonal entries  in this basis. Of course,  acts on this space as well, and so  has trace zero on this space, thus  and so  (here we use the characteristic zero nature of ). From this we see that  fixes . If we let  be an eigenvector of  on  (which exists from the Jordan decomposition of ), we conclude that  is a simultaneous eigenvector of  as required.

The claim (ii) follows from (i) much as in Engel’s theorem. 

**— 3. Characterising semisimplicity —**

The objective of this section will be to prove Theorem 1.

Let  be an concrete Lie algebra, and  be an element of . Then the components  of  need not lie in . However they behave “as if” they lie in  for the purposes of taking Lie brackets, in the following sense:

**Lemma 5** Let  and let  have Jordan decomposition . Then ,  and .

*Proof:* As  and  are semisimple and nilpotent on  and commute with each other,  and  are semisimple and nilpotent on  and also commute with each other (this can for instance by using Lie’s theorem (or the Jordan normal form) to place  in upper triangular form and computing everything explicitly). Thus  is the Jordan-Chevalley decomposition of , and in particular  for some polynomial  with zero constant coefficient. Since  maps  to the subalgebra , we conclude that  does also, thus  as required. Similarly for  and  (note that ). 

We can now use this (together with Engel’s theorem and the test (7) for nilpotency) to obtain a part of Theorem 1:

**Proposition 6** Let  be a simple Lie algebra. Then the Killing form  is non-degenerate.

*Proof:* As  is simple, its center  is trivial, so by (6)  is isomorphic to . In particular we may assume that  is a concrete Lie algebra, thus  for some vector space .

Suppose for contradiction that  is degenerate. Using the skew-adjointness identity



for all  (which comes from the cyclic properties of trace), we see that the kernel  is a non-trivial ideal of , and is thus all of  as  is simple. Thus  for all .

Now let . By Lemma 5,  acts by Lie bracket on  and so one can define . We now consider the quantity



We can rearrange this as



By Lemma 5, , so this is equal to



and so



for all . On the other hand,  is an ideal of ; as  is simple, we must thus have  (i.e.  is perfect). As , we conclude that



From (7) we conclude that  is nilpotent for every . By Engel’s theorem, this implies that , and hence , is nilpotent; but  is simple, giving the desired contradiction. 

**Corollary 7** Let  be a simple ideal of a Lie algebra . Then  is complemented by another ideal  of  (thus  and ), with  isomorphic to the direct sum .

*Proof:* The adjoint action of  restricts to the ideal  and gives a restricted Killing form



By Proposition 6, this bilinear form is non-degenerate on , so the orthogonal complement



is a complementary subspace to . It can be verified to also be an ideal. Since  lies in both  and , we see that , and so  is isomorphic to  as claimed. 

Now we can prove Theorem 1. We first observe that (i) trivially implies (ii); conversely, if  has a non-trivial solvable ideal , then every element of the derived series of  is also an ideal of , and in particular  will have a non-trivial abelian ideal. Thus (i) and (ii) are equivalent.

Now we show that (i) implies (iv), which we do by induction on the dimension of . Of course we may assume  is non-trivial. Let  be a non-trivial ideal of  of minimal dimension. If  then  is simple (note that it cannot be abelian as  is non-trivial and semisimple) and we are done. If  is strictly smaller than , then it also has no non-trivial solvable ideals (because the radical of  is a [characteristic](http://en.wikipedia.org/wiki/Characteristic_subgroup) subalgebra of  and is thus an ideal in ) and so by induction is isomorphic to the direct sum of simple Lie algebras; as  was minimal, we conclude that  is itself simple. By Corollary 7,  then splits as the direct sum of  and a semisimple Lie algebra of strictly smaller dimension, and the claim follows from the induction hypothesis.

From Proposition 6 we see that (iv) implies (iii), so to finish the proof of Theorem 1 it suffices to show that (iii) implies (ii). Indeed, if  has a non-trivial abelian ideal , then for any  and ,  annihilates  and also has range in , hence has trace zero, so  is -orthogonal to , giving the degeneracy of the Killing form.

**Remark 1** Similar methods also give the *Cartan solvability criterion*: a Lie algebra  is solvable if and only if  is orthogonal to  with respect to the Killing form. Indeed, the “only if” part follows easily from Lie’s theorem, while for the “if” part one can adapt the proof of Proposition 6 to show that if  is orthogonal to , then every element of  is nilpotent, hence by Engel’s theorem  is nilpotent, and so from the short exact sequence (6) we see that  is nilpotent, and hence  is solvable.

**Remark 2** The decomposition of a semisimple Lie algebra as the direct sum of simple Lie algebras is unique up to isomorphism and permutation. Indeed, suppose that  is isomorphic to  for some simple . We project each  to  and observe from simplicity that these projections must either be zero or isomorphisms (cf. [Schur’s lemma](http://en.wikipedia.org/wiki/Schur%27s_lemma)). For fixed , there must be at least one  for which the projection is an isomorphism (otherwise  could not generate all of ); on the other hand, as any two  commute with each other in the direct sum, and  is nonabelian, there is at most one  for which the projection is an isomorphism. This gives the required identification of the  and  up to isomorphism and permutation.

**Remark 3** One can also establish complete reducibility by using the [Weyl unitary trick](http://en.wikipedia.org/wiki/Unitarian_trick), in which one first creates a real compact Lie group whose Lie algebra is a real form of the complex Lie algebra being studied, and then uses the complete reducibility of actions of compact groups. This also gives an alternate way to establish Theorem 32 in the appendix.

Semisimple Lie algebras have a number of important non-degeneracy properties. For instance, they have no non-trivial outer automorphisms (at the infinitesimal level, at least):

**Lemma 8** Let  be a semisimple Lie algebra. Then every derivation  on  is inner, thus  for some .

*Proof:* From the identity  we see that  is an ideal in . The trace form  on  restricts to the Killing form on , which is non-degenerate.

Suppose for contradiction that  is not all of , then there is a non-trivial derivation  which is trace-form orthogonal to , thus  is trace-orthogonal to  for all , so that  is trace-orthogonal to  for all . As  is non-degenerate, we conclude that  for all , and so  is trivial, a contradiction. 

This fact, combined with the complete reducibility of -modules (a fact which we will prove in an appendix) implies that the Jordan decomposition preserves concrete semisimple Lie algebras:

**Corollary 9** Let  be a concrete semisimple Lie algebra, and let . Then  also lie in .

*Proof:* By Theorem 1,  is the direct sum of commuting simple algebras. It is easy to see that if  commute then the Jordan decomposition of  arises from the sum of the Jordan decompositions of  and  separately, so we may assume without loss of generality that  is simple.

Observe that if  splits as the direct sum  of two -invariant subspaces (so that  can be viewed as a subalgebra of , and the elements of  can be viewed as being block-diagonal in a suitable basis of ), then the claim for  follows from that of  and . So by an induction on dimension, it suffices to establish the claim under the hypothesis that  is [indecomposable](http://en.wikipedia.org/wiki/Indecomposable_module), in that it cannot be expressed as the direct sum of two non-trivial invariant subspaces.

In the appendix we will show that every invariant subspace  of  is *complemented* in that one can write  for some invariant subspace . Assuming this fact, it suffices to establish the claim in the case that  is *irreducible*, in the sense that it contains no proper invariant subspaces.

By Lemma 7, the operation  is a derivation on , thus there exists  such that  for all , thus  centralises . By [Schur’s lemma](http://en.wikipedia.org/wiki/Schur%27s_lemma) and the hypothesis of irreducibility, we conclude that  is a multiple of a constant . Onthe other hand, every element of  has trace zero since ; in particular,  and  have trace zero, and so  has trace zero. But this trace is just , so we conclude that  and the claim follows. 

This allows us to make the Jordan decomposition universal for semisimple algebras:

**Lemma 10 (Semisimple Jordan decomposition)** Let  be a semisimple Lie algebra, and let . Then we have a unique decomposition  in  such that  and  for every representation  of .

*Proof:* As the adjoint representation is faithful we may assume without loss of generality that  is a concrete algebra, thus . The uniqueness is then clear by taking  to be the identity. To obtain existence, we take  to be the concrete Jordan decomposition. We need to verify  and  for any representation . The adjoint actions of  and  on  commute and are semisimple and nilpotent respectively and so



in  (cf. the proof of Lemma 5). A similar argument (applying Corollary 9 to , which is isomorphic to a quotient of  and is thus semisimple, to keep  in ) gives



Since the adjoint representation of the semisimple algebra  is faithful, the claim follows. 

One can also show that ,  commute with each other and with the centraliser  of  by using the faithful nature of the adjoint representation for semisimple algebras, though we will not need these facts here. Using this lemma we have a well-defined notion of an element  of a semisimple algebra  being semisimple (resp. nilpotent), namely that  or . Lemma 10 then implies that any representation of a semisimple element of  is again semisimple, and any representation of a nilpotent element of  is again nilpotent. This apparently innocuous statement relies heavily on the semisimple nature of ; note for instance that the representation



of the non-semisimple algebra  into  takes semisimple elements to nilpotent ones.

**— 4. Cartan subalgebras —**

While simple Lie algebras do not have any non-trivial ideals, they do have some very useful subalgebras known as [Cartan subalgebras](http://en.wikipedia.org/wiki/Cartan_subalgebra) which will eventually turn out to be abelian and which can be used to dramatically clarify the structure of the rest of the algebra.

We need some definitions. An element  of  is said to be [regular](http://en.wikipedia.org/wiki/Regular_element_of_a_Lie_algebra) if its generalised null space



has minimal dimension. A *Cartan subalgebra* of  is a nilpotent subalgebra  of  which is its own [normaliser](http://en.wikipedia.org/wiki/Centralizer_and_normalizer), thus  is equal to . From the polynomial nature of the Lie algebra operations (and the [Noetherian nature of algebraic geometry](http://en.wikipedia.org/wiki/Noetherian_topological_space)) we see that the regular elements of  are generic (i.e. they form a non-empty [Zariski-open](http://en.wikipedia.org/wiki/Zariski_topology) subset of ).

**Example 1** In , the regular elements consist of the semisimple elements with distinct eigenvalues. Fixing a basis for , the space of elements of  that are diagonalised by that basis form a Cartan subalgebra of .

Cartan algebras always exist, and can be constructed as generalised null spaces of regular elements:

**Proposition 11 (Existence of Cartan subalgebras)** Let  be an abstract Lie algebra. If  is regular, then the generalised null space  of  is a Cartan subalgebra.

*Proof:* Suppose that  is not nilpotent, then by Engel’s theorem the adjoint action of at least one element of  on  is not nilpotent. By the polynomial nature of the Lie algebra operations, we conclude that the adjoint action of a generic element of  on  is not nilpotent.

The action of  on  is non-singular, so the action of generic elements of  on  is also non-singular. Thus we can find  such that  is not nilpotent on  and not singular on . From this we see that  is a proper subspace of , contradicting the regularity of . Thus  is nilpotent.

Finally, we show that  is its own normaliser. Suppose that  normalises , then . But  is the generalised null space of , and so  as required.

Furthermore, all Cartan algebras arise as generalised null spaces:

**Proposition 12 (Cartans are null spaces)** Let  be an abstract Lie algebra, and let  be a Cartan subalgebra. Let



be the generalised null space of . Then . Furthermore, for generic , one has



*Proof:* As  is nilpotent, we certainly have . Now, for any ,  acts nilpotently on both  and  and hence on . By Engel’s theorem, we can thus find  that is annihilated by the adjoint action of ; pulling back to , we conclude that the normaliser of  is strictly larger than , contradicting the hypothesis that  is a Cartan subalgebra. This shows that .

Now let  be generic, then  has minimal dimension amongst . Let  be arbitrary. Then for any scalar ,  acts on  and on  and hence on . This action is invertible when , and hence is also invertible for generic ; thus for generic , . By minimality we conclude that , so  is nilpotent on  for generic , and thus for all . In particular  is nilpotent on  for any , thus . Since , we obtain  as required. 

**Corollary 13 (Cartans are conjugate)** Let  be a Lie algebra, and let  be a Cartan algebra. Then for generic ,  is conjugate to  by an inner automorphism of  (i.e. an element of the algebraic group generated by  for ). In particular, any two Cartan subalgebras are conjugate to each other by an inner automorphism.

*Proof:* Let  be the set of  with , then  is a Zariski open dense subset of  by Proposition 12. Then let  be the collection of  that are conjugate to an , then  is a algebraically constructible subset of . For , observe that  and  span , since , and so by the [inverse function theorem](http://en.wikipedia.org/wiki/Inverse_function_theorem), a (topological) neighbourhood of  is contained in . This implies that  is Zariski dense, and the claim follows. 

In the case of semisimple algebras, the Cartan structure is particularly clean:

**Proposition 14** Let  be a semisimple Lie algebra. Then every Cartan subalgebra  is abelian, and  is non-degenerate on .

The dimension of the Cartan algebra of a semisimple Lie algebra is known as the *rank* of the algebra.

*Proof:* The nilpotent algebra  acts via the adjoint action on , and by Lie’s theorem this action can be made upper triangular. From this it is not difficult to obtain a decomposition



for some finite set , where  are the generalised eigenspaces



From the Jacobi identity (2) we see that . Among other things, this shows that  has ad-trace zero for any non-zero , and hence  are -orthogonal if . In particular,  is -orthogonal to . By Theorem 1,  is non-degenerate on , and thus also non-degenerate on ; by Proposition 12,  is thus non-degenerate on . But by Lie’s theorem, we can find a basis for which  consists of upper-triangular matrices in the adjoint representation of , so that  is strictly upper-triangular and thus -orthogonal to . As  is non-degenerate on , this forces  to be trivial, as required. 

We now use the semisimple Jordan decomposition (Lemma 10) to obtain a further non-degeneracy property of the Cartan subalgebras of semisimple algebras:

**Proposition 15** Let  be a semisimple Lie algebra. Then every Cartan subalgebra  consists entirely of semisimple elements.

*Proof:* Let , then (by the abelian nature of )  annihilates ; as  is a polynomial in  with zero constant coefficient,  annihilates  as well; thus  normalises  and thus also lies in  as  is Cartan. If , then  commutes with  and so  commutes with . As the latter is nilpotent, we conclude that  is nilpotent and thus has trace zero. Thus  is -orthogonal to  and thus vanishes since the Killing form is non-degenerate on . Thus every element of  is semisimple as required. 

**— 5.  representations —**

To proceed further, we now need to perform some computations on a very specific Lie algebra, the special linear algebra  of  complex matrices with zero trace. This is a three-dimensional concrete Lie algebra, spanned by the three generators



which obey the commutation relations



Conversely, any abstract three-dimensional Lie algebra generated by  with relations (8) is clearly isomorphic to . One can check that this is a simple Lie algebra, with the one-dimensional space generated by  being a Cartan subalgebra.

Now we classify by hand the representations  of . Observe that  acts infinitesimally on  by the differential operators (or vector fields)



In particular, we see that for each natural number , the space  of homogeneous polynomials in two variables  of degree  has a representation ; if we give this space the basis  for , the action is then described by the formulae



for . From these formulae it is also easy to see that these representations are irreducible in the sense that the  have no non-trivial -invariant subspaces.

Conversely, these representations (and their direct sums) describe (up to isomorphism) all of the representations of :

**Theorem 16 (Representations of )** Any representation  is isomorphic to the direct sum of finitely many of the representations .

Here of course the direct sum  of two representations ,  is defined as , and two representations ,  are isomorphic if there is an invertible linear map  such that  for all .

*Proof:* By induction we may assume that  is non-trivial, the claim has already been proven for any smaller dimensional spaces than .

As  is semisimple,  is semisimple by Lemma 10, and so we can split  into the direct sum



of eigenspaces of  for some finite .

From (8) we have the raising law



and the lowering law



As  is finite, we may find a “highest weight”  with the property that , thus  annihilates  by the raising law. We will use the basic strategy of starting from the highest weight space and applying lowering operators to discover one of the irreducible components of the representation.

From (8) one has



and so from induction and the lowering law we see that



for all natural numbers  and all . If  is never zero, this creates an infinite sequence  of non-trivial eigenspaces, which is absurd, so we have  for some natural number , thus . If we then let



then we see that  is invariant under , , and , and thus -invariant; also if for each  we let  be the set of all  such that  is never a non-zero element of  then we see that



is also -invariant, and furthermore that  and  are complementary subspaces in . Applying the induction hypothesis, we are done unless , but then by splitting  into one-dimensional spaces and applying the lowering operators, we see that we reduce to the case that  is one-dimensional. But if one then lets  be a generator of  and recursively defines  by



one then checks using (10) that  is isomorphic to , and the claim follows. 

**Remark 4** Theorem 16 shows that all representations of  are *completely reducible* in that they can be decomposed as the direct sum of irreducible representations. In fact, all representations of semisimple Lie algebras are completely reducible; this can be proven by a variant of the above arguments (in combination with the analysis of weights given below), and can also be proven by the unitary trick, or by analysing the action of [Casimir elements](http://en.wikipedia.org/wiki/Casimir_invariant) of the universal enveloping algebra of , as done in the Appendix.

**— 6. Root spaces —**

Now we use the  theory to analyse more general semisimple algebras.

Let  be a semisimple Lie algebra, and let  be a Cartan algebra, then by Proposition 14  is abelian and acts in a semisimple fashion on , and by Proposition 12  is its own null space  in the weight decomposition of , thus we have the [Cartan decomposition](http://en.wikipedia.org/wiki/Cartan_decomposition)



as vector spaces (not as Lie algebras) where  is a finite subset of  (known as the set of *roots*) and  is the non-trivial eigenspace



**Example 2** A key example to keep in mind is when  is the Lie algebra of  matrices of trace zero. An explicit computation using the Killing form and Theorem 1 shows that this algebra is semisimple; in fact it is simple, but we will not show this yet. The space  of diagonal matrices of trace zero can then be verified to be a Cartan algebra; it can be identified with the space  of complex -tuples summing to zero, and using the usual Hermitian inner product on  we can also identify  with . The roots are then of the form  for distinct , where  is the standard basis for , with  being the one-dimensional space of matrices that are vanishing except possibly at the  coefficient.

From the Jacobi identity (2) we see that the Lie bracket acts additively on the weights, thus



for all . Taking traces, we conclude that



whenever . As  is non-degenerate, we conclude that if  is non-trivial, then  must also be non-trivial, thus  is symmetric around the origin.

We also claim that  spans  as a vector space. For if this were not the case, then there would be a non-trivial  that is annihilated by , which by (11) implies that  annihilates all of the  and is thus central, contradicting the semisimplicity of .

From Proposition 14,  is non-degenerate on . Thus, for each root , there is a corresponding non-zero element  of  such that  for all . If we let , and , we have









and thus by the non-degeneracy of  on  we obtain the useful formula



for  and .

As  is non-degenerate, we can find  and  with  (which can be found as  is non-degenerate). We divide into two cases depending on whether  vanishes or not. If  vanishes, then  is non-trivial but commutes with  and , and so  generate a solvable algebra. By Lie’s theorem, this algebra is upper-triangular in some basis, and so  is nilpotent, hence  is nilpotent; but by Proposition 15  is also semisimple, contradicting the non-zero nature of  (and the semisimple nature of ). Thus  is non-vanishing. If we then scale  so that , where  is the [co-root](http://en.wikipedia.org/wiki/Co-root#Dual_root_system_and_coroots) of , defined as the element of  given by the formula



so that



then  obey the relations (8) and thus generate a copy of , rather than a solvable algebra. The representation theory of  can then be applied to the space



where . By (19), this space is invariant with respect to  and  and hence to the copy of , and by (11), (14) each  is the weight space of  of weight  for each . By Theorem 16, we conclude that the set  consists of integers. On the other hand, from (13) we see that any copy of the representation  with  a positive even integer must have its  weight space contained in the span of , and so there is only one such representation in (15). As  already give a copy of  in (15), there are no other copies of  with  positive even, thus we have that  is one-dimensional and that the only even multiples of  in  are . In particular,  whenever , which also implies that  whenever . Returning to Theorem 16, we conclude that the set  contains no odd integers, and so  and  are the only multiples of  in .

Next, let  be any non-zero element of  orthogonal to  with respect to the inner product  of  that is dual to the restriction of the Killing form to , and consider the space



where



By (19), this is again an -invariant space, and by (11), (14) each  is the weight space of  of weight . From Theorem 16 we see that  is an arithmetic progression  of spacing ; in particular,  is symmetric around the origin and consists only of integers. This implies that the set  is symmetric with respect to reflection across the hyperplane that is orthogonal to , and also implies that



for all roots .

We summarise the various geometric properties of  as follows:

**Proposition 17 (Root systems)** Let  be a semisimple Lie algebra, let  be a Cartan subalgebra, and let  be the inner product on  that is dual to the Killing form restricted to . Let  be the set of roots. Then:

* (i)  does not contain zero.
* (ii) If  is a root, then  is symmetric with respect to the reflection operation  across the hyperplane orthogonal to ; in particular,  is also a root.
* (iii) If  is a root, then no multiple of  other than  are roots.
* (iv) If  are roots, then  is an integer or half-integer. Equivalently,  for some integer .
* (v)  spans .

A set of vectors  obeying the above axioms (i)-(v) is known as a [root system](http://en.wikipedia.org/wiki/Root_system) on  (viewed as a finite dimensional complex Hilbert space with the inner product ).

**Remark 5** A short calculation reveals the remarkable fact that if  is a root system, then the associated system of co-roots  is also a root system. This is one of the starting points for the deep phenomenon of [Langlands duality](http://en.wikipedia.org/wiki/Langlands_dual), which we will not discuss here.

When  is simple, one can impose a useful additional axiom on . Say that a root system  is *irreducible* if  cannot be covered by the union  of two orthogonal proper subspaces of .

**Lemma 18** If  is a simple Lie algebra, then the root system of  is irreducible.

*Proof:* If  can be covered by two orthogonal subspaces , then if we consider the subspace of 



where we use the inner product  to identify  with  and thus  with a subspace of  (thus for instance this identifies  with ), then one can check using (19) and (13) that this is a proper ideal of , contradicting simplicity. 

It is easy to see that every root system is expressible as the union of irreducible root systems (on orthogonal subspaces of ). As it turns out, the irreducible root systems are completely classified, with the complete list of root systems (up to isomorphism) being described in terms of the [Dynkin diagrams](http://en.wikipedia.org/wiki/Dynkin_diagram)  briefly mentioned in Theorem 2. We will now turn to this classification in the next section, and then use root systems to recover the Lie algebra.

**— 7. Classification of root systems —**

In this section we classify all the irreducible root systems  on a finite dimensional complex Hilbert space , up to Hilbert space isometry. Of course, we may take  to be a standard complex Hilbert space  without loss of generality. The arguments here are purely elementary, proceeding purely from the root system axioms rather than from any Lie algebra theory.

Actually, we can quickly pass from the complex setting to the real setting. By axiom (v),  contains a basis  of ; by axiom (iv), the inner products between these basis vectors are real, as are the inner products between any other root and a basis root. From this we see that  lies in the *real* vector space spanned by the basis roots, so by a change of basis we may assume without loss of generality that .

Henceforth  is assumed to lie in . From two applications of (iv) we see that for any two roots , the expression



lies in ; but it is also equal to , and hence



for all roots . Analysing these cases further using (iv) again, we conclude that there are only a restricted range of options for a pair of roots :

**Lemma 19** Let  be roots. Then one of the following occurs:

* (0)  and  are orthogonal.
* (1/4)  have the same length and subtend an angle of  or .
* (1/2)  has  times the length of  or vice versa, and  subtend an angle of  or .
* (3/4)  has  times the length of  or vice versa, and  subtend an angle of  or .
* (1) .

We next record a useful corollary of Lemma 19 (and axiom (ii)):

**Corollary 20** Let  be roots. If  subtend an acute angle, then  and  are also roots, but  is not a root. Equivalently, if  subtend an obtuse angle, then  is a root, but  and  are not roots.

This follows from a routine case analysis and is omitted.

We can leverage Corollary 20 as follows. Call an element  of  *regular* if it is not orthogonal to any root, thus generic elements of  are regular. Given a regular element , let  denote the roots  which are *-positive* in the sense that their inner product with  is positive; thus  is partitioned into  and . We will abbreviate -positive as *positive* if  is understood from context. Call a positive root  a *-simple root* (or *simple root* for short) if it cannot be written as the sum of two positive roots. Clearly every positive root is then a linear combination of simple roots with natural number coefficients. By Corollary 20, two simple roots cannot subtend an acute angle, and so any two distinct simple roots subtend a right or obtuse angle.

**Example 3** Using the root system  of  discussed previously, if one takes  to be any vector in  with decreasing coefficients, then the positive roots are those roots  with , and the simple roots are the roots  for .

Define an *admissible configuration* to be a collection of unit vectors in  in a open half-space  with the property that any two vectors in this collection form an angle of , , , or , and call the configuration *irreducible* if it cannot be decomposed into two non-empty orthogonal subsets. From Lemma 19 and the above discussion we see that the unit vectors  associated to the simple roots are an admissible configuration. They are also irreducible, for if the simple roots partition into two orthogonal sets then it is not hard to show (using Corollary 20) that all positive roots lie in the span of one of these two sets, contradicting irreducibility of the root system.

We can say quite a bit about admissible configurations; the fact that the vectors in the system always subtend right or obtuse angles, combined with the half-space restriction, is quite limiting (basically because this information can be in violation of inequalities such as the [Bessel inequality](http://en.wikipedia.org/wiki/Bessel_inequality), or the positive (semi-)definiteness  of the [Gram matrix](http://en.wikipedia.org/wiki/Gram_matrix)). We begin with an assertion of linear independence:

**Lemma 21** If  is an admissible configuration, then it is linearly independent.

Among other things, this shows that the number of simple roots of a semisimple Lie algebra is equal to the rank of that algebra.

*Proof:* Suppose this is not the case, then one has a non-trivial linear constraint



for some positive  and disjoint . But as any two vectors in an admissible configuration subtend a right or obtuse angle, , and thus . But this is not possible as all the  lie in an open half-space. 

Define the [Coxeter diagram](http://en.wikipedia.org/wiki/Coxeter_diagram) of an admissible configuration  to be the graph with vertices , and with any two vertices  connected by an edge of multiplicity , thus two vertices are unconnected if they are orthogonal, connected with a single edge if they subtend an angle of , a double edge if they subtend an angle of , and a triple edge if they subtend an angle of . The irreducibility of a configuration is equivalent to the connectedness of a Coxeter diagram. Note that the Coxeter diagram describes all the inner products between the  and thus describes the  up to an orthogonal transformation (as can be seen for instance by applying the Gram-Schmidt process).

**Lemma 22** The Coxeter diagram of an admissible configuration is [acyclic](http://en.wikipedia.org/wiki/Cycle_(graph_theory)) (ignoring multiplicity of edges). In particular, the Coxeter diagram of an irreducible admissible configuration is a [tree](http://en.wikipedia.org/wiki/Tree_(graph_theory)).

*Proof:* Suppose for contradiction that the Coxeter diagram contains a cycle , we see that  for  (with the convention ) and  for all other . This implies that , which contradicts the linear independence of the . 

**Lemma 23** Any vertex in the Coxeter diagram has degree at most three (counting multiplicity).

*Proof:* Let  be a vertex which is adjacent to some other vertices , which are then an orthonormal system. By Bessel’s inequality (and linear independence) one has



But from construction of the Coxeter diagram we have  for each , where  is the multiplicity of the edge connecting  and . The claim follows. 

We can also contract simple edges:

**Lemma 24** If  is an admissible configuration with  joined by a single edge, then the configuration formed from  by replacing  with the single vertex  is again an admissible configuration, with the resulting Coxeter diagram formed from the original Coxeter diagram by deleting the edge between  and  and then identifying  together.

This follows easily from acyclicity and direct computation.

By Lemma 23 and Lemma 24, the Coxeter diagram can never form a vertex of degree three no matter how many simple edges are contracted. From this we can easily show that connected Coxeter diagrams must have one of the following shapes:

* :  vertices joined in a chain of simple edges;
* :  vertices joined in a chain of edges, one of which is a double edge and all others are simple edges;
* : three chains of simple edges emenating from a common vertex (forming a “Y” shape), connecting  vertices in all;
* : Two vertices joined by a triple edge.

We can cut down the  and  cases further:

**Lemma 25** The Coxeter diagram of an admissible configuration cannot contain as a subgraph

* (a) A chain of four edges, with one of the interior edges a double edge;
* (b) Three chains of two simple edges each, emenating from a common vertex;
* (c) Three chains of simple edges of length  respectively, emenating from a common vertex.

*Proof:* To exclude (a), suppose for contradiction that we have two chains  and  of simple edges, with  joined by a double edge. Writing  and , one computes that  are unit vectors with inner product , implying that  are parallel, contradicting linear independence.

To exclude (b), suppose that we have three chains , ,  of simple edges joined at . Then the vectors  are an orthonormal system that each have an inner product of  each with . Comparing this with Bessel’s inequality we conclude that  lies in the span of , contradicting linear independence.

Finally, to exclude (c), suppose we have three chains , ,  of simple edges joined at . Writing , , , we compute that  are an orthonormal system that have inner products of  respectively with . As , this forces  to lie in the span of , again contradicting linear independence. 

We remark that one could also obtain the required contradictions in the above proof by verifying in all three cases that the [Gram matrix](http://en.wikipedia.org/wiki/Gramian_matrix) of the subconfiguration has determinant zero.

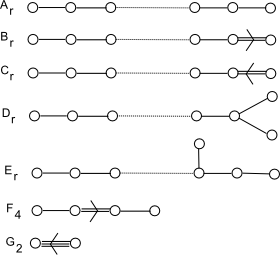
**Corollary 26** The Coxeter diagram of an irreducible admissible configuration must take one of the following forms:

* :  vertices joined in a chain of simple edges for some ;
* :  vertices joined in a chain of edges for some , with one boundary edge being a double edge and all other edges simple;
* : Three chains of simple edges of length  respectively for some , emenating from a single vertex;
* : Three chains of simple edges of length  respectively for some , emenating from a single vertex;
* : Four vertices joined in a chain of edges, with the middle edge being a double edge and the other two edges simple;
* : Two vertices joined by a triple edge.

Now we return to root systems. Fixing a regular , we define the [Dynkin diagram](http://en.wikipedia.org/wiki/Dynkin_diagram) to be the Coxeter diagram associated to the (unit vectors of the) simple roots, except that we orient the double or triple edges to point from the longer root to the shorter root. (Note from Lemma 19 that we know exactly what the ratio between lengths is in these cases; in particular, the Dynkin diagram describes the root system up to a unitary transformation and dilation.) We conclude

**Corollary 27** The Dynkin diagram of an irreducible root system must take one of the following forms:

* :  vertices joined in a chain of simple edges for some ;
* :  vertices joined in a chain of edges for some , with one boundary edge being a double edge (pointing outward) and all other edges simple;
* :  vertices joined in a chain of edges for some , with one boundary edge being a double edge (pointing inward) and all other edges simple;
* : Three chains of simple edges of length  respectively for some , emenating from a single vertex;
* : Three chains of simple edges of length  respectively for some , emenating from a single vertex;
* : Four vertices joined in a chain of edges, with the middle edge being a double (oriented) edge and the other two edges simple;
* : Two vertices joined by a triple (oriented) edge.



This describes (up to isomorphism and dilation) the simple roots:

* : The simple roots take the form  for  in the space  of vectors whose coefficients sum to zero;
* : The simple roots take the form  for  and also  in .
* : The simple roots take the form  for  and also  in .
* : The simple roots take the form  for  and also  in .
* : The simple roots take the form  for  and also  and  in .
* : This system is obtained from  by deleting the first one or two simple roots (and cutting down  appropriately)
* : The simple roots take the form  for  and also  and  in .
* : The simple roots take the form ,  in .

**Remark 6** A slightly different way to reach the classification is to replace the Dynkin diagram by the *extended Dynkin diagram* in which one also adds the maximal negative root in addition to the simple roots; this breaks the linear independence, but one can then label each vertex by the coefficient in the linear combination needed to make the roots sum to zero, and one can then analyse these multiplicities to classify the possible diagrams and thence the root systems.

Now we show how the simple roots can be used to recover the entire root system. Define the [Weyl group](http://en.wikipedia.org/wiki/Weyl_group)  to be the group generated by all the reflections  coming from all the roots ; as the roots span  and obey axiom (ii), the Weyl group acts faithfully on the finite set  and is thus itself finite.

**Lemma 28** Let  be regular, and let  be any element of . Then there exists  such that  for all -simple roots  (or equivalently, for all -positive roots ). In particular, if  is regular, then , so that all -simple roots are -simple and vice versa.

Furthermore, every root can be mapped by an element of  to an -simple root.

Finally,  is generated by the reflections  coming from the -simple roots .

*Proof:* Let  be a simple root. The action of the reflection  maps  to , and maps all other simple roots  to  for some non-negative  (since  subtend a right or obtuse angle). In particular, we see that  maps all positive roots other than  to positive roots, and hence (as  is an involution)



In particular, if we define , then



for all simple roots .

Let  be the subgroup of  generated by the  for the simple roots , and choose  to maximise . Then from (17) we have , giving the first claim. Since every root  is -simple for some regular  (by selecting  to very nearly be orthogonal to ), we conclude that every root can be mapped by an element of  to a -simple root in , giving the second claim. Thus for any root ,  is conjugate in  to a reflection  for a -simple root , so  lies in  and so , giving the final claim. 

**Remark 7** The set of all  for which  is known as the [Weyl chamber](http://en.wikipedia.org/wiki/Weyl_group) associated to ; this is an open polyhedral cone in , and the above lemma shows that it is the interior of a fundamental domain of the action of the Weyl group. In the case of the special linear group, the standard Weyl chamber (in  now instead of ) would be the set of vectors  with decreasing coefficients.

From the above lemma we can reconstruct the root system from the simple roots by using the reflections  associated to the simple roots to generate the Weyl group , and then applying the Weyl group to the simple roots to recover all the roots. Note that the lemma also shows that the set of -simple roots and -simple roots are isomorphic for any regular , so that the Dynkin diagram is indeed independent (up to isomorphism) of the choice of regular element  as claimed earlier. We have thus in principle described the irreducible root systems (up to isomorphism) as coming from the Dynkin diagrams ; see for instance the [Wikipedia page on root systems](http://en.wikipedia.org/wiki/Root_system) for explicit descriptions of all of these. With these explicit descriptions one can verify that all of these systems are indeed irreducible root systems.

**— 8. Chevalley bases —**

Now that we have described root systems, we use them to reconstruct Lie algebras. We first begin with an abstract uniqueness result that shows that a simple Lie algebra is determined up to isomorphism by its root system.

**Theorem 29 (Root system uniquely determines a simple Lie algebra)** Let  be simple Lie algebras with Cartan subalgebras ,  and root systems , . Suppose that one can identify  with  as vector spaces in such a way that the root systems agree: . Then the identification between  and  can be extended to an identification of  and  as Lie algebras.

*Proof:* First we note from (11) and the identification  that the Killing forms on  and  agree, so we will identify  as Hilbert spaces, not just as vector spaces.

The strategy will be exploit a Lie algebra version of the [Goursat lemma](http://en.wikipedia.org/wiki/Goursat_lemma) (or the [Schur lemma](http://en.wikipedia.org/wiki/Schur_lemma)), finding a sufficiently “non-degenerate” subalgebra  of  and using the simple nature of  and  to show that this subalgebra is the graph of an isomorphism from  to . This strategy will follow the same general strategy used in Theorem 16, namely to start with a “highest weight” space and apply lowering operators to discover the required graph.

We turn to the details. Pick a regular element  of , so that one has a notion of a positive root. For every simple root , we select non-zero elements , of  respectively such that



where  is the co-root of ; similarly select  in , and set  and . Let  be the subalgebra of  generated by the  and . It is not hard to see that the  generate  as a Lie algebra, so  surjects onto ; similarly  surjects onto .

Let  be a maximal root, that is to say a root such that  is not a root for any positive ; such a root always exists. (It is in fact unique, though we will not need this fact here.) Then we have one-dimensional spaces  and , and thus a two-dimensional subspace  in . Inside this subspace, we select a one-dimensional subspace  which is not equal to  or ; in particular,  is not contained in  or .

Let  be the subspace of  generated by  and the adjoint action of the lowering operators , thus it is spanned by elements of the form



for simple roots  and . Then  contains  and is thus not contained in ; because (19) only involves lowering operators, we also see that  does not contain any other element of  other than . In particular,  is not all of .

Clearly  is closed under the adjoint action of the lowering operators . We claim that it is also closed under the adjoint action of the raising operators . To see this, first observe that  commute when  are distinct simple roots, because  cannot be a root (since this would make one of  non-simple). Next, from (18) we see that  acts as a scalar on any element of the form (19), while from the maximality of  we see that  annihilates . From this the claim easily follows.

As  is closed under the adjoint action of both the  and the , we have . Projecting onto , we see that the projection of  is an ideal of , and is hence  or  as  is simple. As  is not contained in , we see that  surjects onto ; similarly it surjects onto . An analogous argument shows that the intersection of  with  is either  or ; the latter would force  by the surjective projection onto , which was already ruled out. Thus  has trivial intersection with , and similarly with , and is thus a graph. Such a graph cannot be an ideal of , so that . As  was a subalgebra that surjected onto both  and , we conclude by arguing as before that  is also a graph; as  is a Lie algebra, the graph is that of a Lie algebra isomorphism by the Lie algebra closed graph theorem (see [this previous blog post](https://terrytao.wordpress.com/2012/11/20/the-closed-graph-theorem-in-various-categories/)). Since , we see that  restricts to the graph of the identity on , and the claim follows. 

**Remark 8** The above arguments show that every root can be obtained from the maximal root by iteratively subtracting off simple roots (while staying in ), which among other things implies that the maximal root is unique. These facts can also be established directly from the axioms of a root system (or from the classification of root systems), but we will not do so here. By using Theorem 29, one can convert graph automorphisms of the Dynkin diagram (e.g. the automorphism sending the  Dynkin diagram to its inverse, or the [triality](http://en.wikipedia.org/wiki/Triality) automorphism that rotates the  diagram) to automorphisms of the Lie algebra; these are important in the theory of twisted [groups of Lie type](http://en.wikipedia.org/wiki/Groups_of_lie_type), and more specifically the [Steinberg groups](http://en.wikipedia.org/wiki/Groups_of_lie_type#Steinberg_groups) and [Suzuki-Ree groups](http://en.wikipedia.org/wiki/Groups_of_lie_type#Suzuki.E2.80.93Ree_groups), but will not be discussed further here.

**Remark 9** In a converse direction, once one establishes that in an irreducible root system  that every root can be obtained from the maximal root by subtracting off simple roots (while staying in ), this shows that any Lie algebra  associated to this system is necessarily simple. Indeed, given any non-trivial ideal  in  and a non-trivial element  of , one locates a minimal element of  in which  has a non-trivial component, then iteratively applies raising operators to then locate a non-trivial element of the root space of the maximal root in ; if one then applies lowering operators one recovers all the other root spaces, so that .

Theorem 29, when combined with the results from previous sections, already gives Theorem 2, but without a fully explicit way to determine the Lie algebras  listed in that theorem (or even to establish whether these systems exist at all). In the case of the [classical Lie algebras](http://en.wikipedia.org/wiki/Classical_Lie_algebra) , one can explicitly describe these algebras in terms of the [special linear algebras](http://en.wikipedia.org/wiki/Special_linear_algebra) , [special orthogonal algebras](http://en.wikipedia.org/wiki/Special_orthogonal_group) , and [symplectic algebras](http://en.wikipedia.org/wiki/Symplectic_group) , but this does not give too much guidance as to how to explicitly describe the *exceptional Lie algebras* . We now turn to the question of how to explicitly describe all the simple Lie algebras in a unified fashion.

Let  be a simple Lie algebra, with Cartan algebra . We view  as a Hilbert space with the Killing form, and then identify this space with its dual . Thus for instance the coroot  of a root  is now given by the simpler formula



Let  be the root system, which is irreducible. As described in Section 6, we have the vector space decomposition



where the spaces  are one-dimensional, thus we can choose a generator  for each , though we have the freedom to multiply each  by a complex constant, which we will take advantage of to perform various normalisations. A basis for algebra  together with the  then form a basis for , known as a *Cartan-Weyl basis* for this Lie algebra. From (11), (20) we have



where  is the quantity



which is always an integer because  is a root system (indeed  takes values in , and form an interesting matrix known as the [Cartan matrix](http://en.wikipedia.org/wiki/Cartan_matrix)).

As discussed in Section 6,  is a multiple of the coroot ; by adjusting  for each pair  we may normalise things so that



for all  (here we use the fact that  to avoid inconsistency). Next, we see from (19) that



if , and



for some complex number  if . By considering the action of  on (16) using Theorem 16 one can verify that  is non-zero; however, its value is not yet fully determined because there is still residual freedom to normalise the . Indeed, one has the freedom to multiply  by any non-zero complex scalar  as long as  (to preserve the normalisation (21)), in which case the structure constant  gets transformed according to the law



However, observe that the combined structure constant  is unchanged by this rescaling. And indeed there is an explicit formula for this quantity:

**Lemma 30** For any roots  with , one has



where  are the string of roots of the form  for integer .

This formula can be confirmed by an explicit computation using Theorem 16 (using, say, the standard basis for  to select , which then fixes  by (21)); we omit the details.

On the other hand, we have the following clever renormalisation trick of Chevalley, exploiting the abstract isomorphism from Theorem 29:

**Lemma 31 (Chevalley normalisation)** There exist choices of  such that



for all roots  with .

*Proof:* We first select  arbitrarily, then we will have



for some non-zero  for all roots . The plan is then to locate coefficients  so that the transformation (23) eliminates all of the  factors.

To do this, observe that we may identify  with itself and  with itself via the negation map  for  and  for . From this and Theorem 29, we may find a Lie algebra isomorphism  that maps  to  on , and thus maps  to  for any root . In particular, we have



for some non-zero coefficients ; from (21) we see in particular that



If we then apply  to (22), we conclude that



when  is a root, so that  takes the special form



If we then select  so that



for all roots  (this is possible thanks to (24)), then the transformation (23) eliminates  as desired. 

From the above two lemmas, we see that we can select a special Cartan-Weyl basis, known as a [Chevalley basis](http://en.wikipedia.org/wiki/Chevalley_basis), such that



whenever  is a root; in particular, the structure constants  are all integers, which is a crucial fact when one wishes to construct Lie algebras and [Chevalley groups](http://en.wikipedia.org/wiki/Chevalley_group#Chevalley_groups) over fields of arbitrary characteristic. This comes very close to fully describing the Lie algebra structure associated to a given Dynkin diagram, except that one still has to select the signs  in (25) so that one actually gets a Lie algebra (i.e. that the Jacobi identity (1) is obeyed). This turns out to be non-trivial; see [this paper of Tits](http://www.ams.org/mathscinet-getitem?mr=214638) for details. (There are other approaches to demonstrate existence of a Lie algebra associated to a given root system; one popular one proceeds using the Chevalley-Serre relations, see e.g. this [text of Serre](http://www.ams.org/mathscinet-getitem?mr=215886). There is still a certain amount of freedom to select the signs, but this ambiguity can be described precisely; see [the book of Carter](http://www.ams.org/mathscinet-getitem?mr=1266626) for details.) Among other things, this construction shows that every root system actually creates a Lie algebra (thus far we have only established uniqueness, not existence), though once one has the classification one could also build a Lie algebra explicitly for each Dynkin diagram by hand (in particular, one can build the simply laced classical Lie algebras  and the maximal simply laced exceptional algebra , and construct the remaining Lie algebras by taking fixed points of suitable involutions; see e.g. [these notes of Borcherds](http://math.berkeley.edu/~theojf/LieQuantumGroups.pdf) et al. for this approach).

**— 9. Appendix: Casimirs and complete reducibility —**

In this appendix we supply a proof of the following fact, used in the proof of Corollary 9:

**Theorem 32 (Weyl’s complete reducibility theorem)** Let  be a simple Lie algebra, and let  be a -invariant subspace of . Then there exists a complementary -invariant subspace  such that .

Among other things, [Weyl’s complete reducibility theorem](http://en.wikipedia.org/wiki/Weyl's_theorem_on_complete_reducibility) shows that every finite-dimensional linear representation of  splits into the direct sum of irreducible representations, which explains the terminology. The claim is also true for semisimple Lie algebras , but we will only need the simple case here, which allows for some minor simplifications to the argument.

The proof of this theorem requires a variant  of the Killing form associated to , defined by the formula



and a certain element of  associated to this form known as the [Casimir operator](http://en.wikipedia.org/wiki/Casimir_element). We first need to establish a variant of Theorem 1:

**Proposition 33** With the hypotheses of Theorem 32,  is non-degenerate.

*Proof:* This is a routine modification of Proposition 6 (one simply omits the use of the adjoint representation). 

Once one establishes non-degeneracy, one can then define the *Casimir operator*  by setting



whenever  is a basis of  and  is its [dual basis](http://en.wikipedia.org/wiki/Dual_basis), thus . It is easy to see that this definition does not depend on the choice of basis, which in turn (by infinitesimally conjugating both bases by an element  of the algebra ) implies that  commutes with every element  of .

On the other hand,  does not vanish entirely. Indeed, taking traces and using (26) we see that



This already gives an important special case of Theorem 32:

**Proposition 34** Theorem 32 is true when  has codimension one and is irreducible.

*Proof:* The Lie algebra  acts on the one-dimensional space ; since  (from the simplicity hypothesis), we conclude that this action is trivial. In other words, each element of  maps  to , so the Casimir operator  does as well. In particular, the trace of  on  is the same as the trace of  on . On the other hand, by Schur’s lemma,  is a constant on ; applying (27), we conclude that this constant is non-zero. Thus  is non-degenerate on , but is not full rank on  as it maps  to . Thus it must have a one-dimensional null-space  which is complementary to . As  commutes with ,  is -invariant, and the claim follows. 

We can then remove the irreducibility hypothesis:

**Proposition 35** Theorem 32 is true when  has codimension one.

We remark that this statement is essentially a reformulation of [Whitehead’s lemma](http://en.wikipedia.org/wiki/Whitehead%27s_lemma_(Lie_algebras)).

*Proof:* We induct on the dimension of  (or ). If  is irreducible then we are already done, so suppose that  has a proper invariant subspace . Then  has codimension one in , so by the induction hypothesis  is complemented by a one-dimensional invariant subspace  of , which lifts to an invariant subspace  of  in which  has codimension one. By the induction hypothesis again,  is complemented by a one-dimensional invariant subspace  in , and it is then easy to see that  also complements  in , and the claim follows. 

Next, we remove the codimension one hypothesis instead:

**Proposition 36** Theorem 32 is true when  is irreducible.

*Proof:* Let  be the space of linear maps  whose restriction to  is a constant multiple of the identity, and let  be the subalgebra of  whose restriction to  vanishes. Then  are -invariant (using the Lie bracket action), and  has codimension one in . Applying Proposition 35 (pushing  forward to , and treating the degenerate case when  vanishes separately) we see that  is complemented by a one-dimensional invariant subspace  of . Thus there exist  that does not lie in , and which commutes with every element of . The kernel  of  is then an invariant complement of  in , and the claim follows. 

Applying the induction argument used to prove Proposition 35, we now obtain Theorem 32 in full generality.